Method of Solution by Separation

Example of Solution by Separation (1)

Solve the initial value problem

$$y'[t] = \frac{t}{1 + y^3}$$

By inspection, we can see that the differential equation is separable.

$t = y'[t](1 + y^3)$

The idea is to force *Mathematica* to set up and solve the equation by mimicing what we do on paper. This example is easy enough to do "in our head".

```
rhs = t
```

```
t
```

```
lhs = (1 + y[t]^3) y'[t]
```

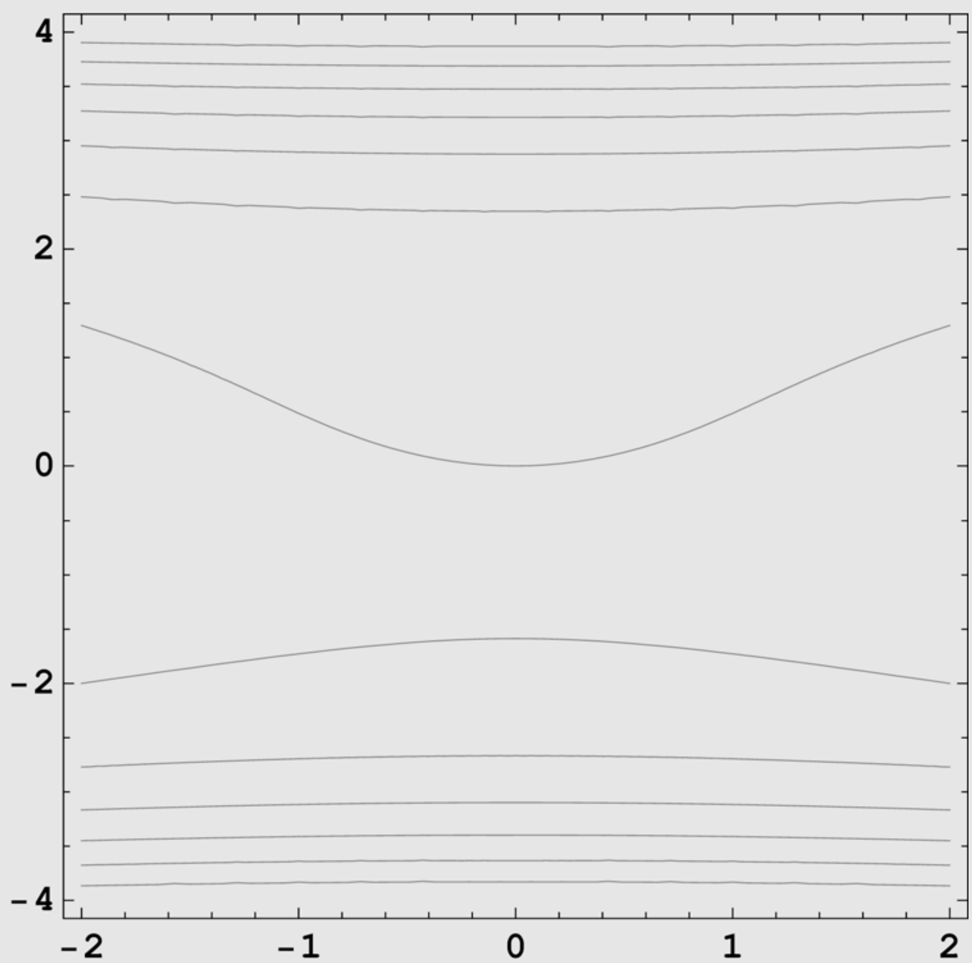
$(1 + y[t]^3)\, y'[t]$

```
implicitsoln = Integrate[lhs, t] == Integrate[rhs,t] +
```

$$y[t] + \frac{y[t]^4}{4} == C + \frac{t^2}{2}$$

Because the left side is a high degree polynomial in y, there is not much hope of explicitly solving for y. Nevertheless, we can look at the solution curves defined implicitly by the implicit solution.

```
cp = ContourPlot[ y + y^4/4 - t^2/2,  {t, -2,2}, {y,-4
ContourShading-> False ]
```

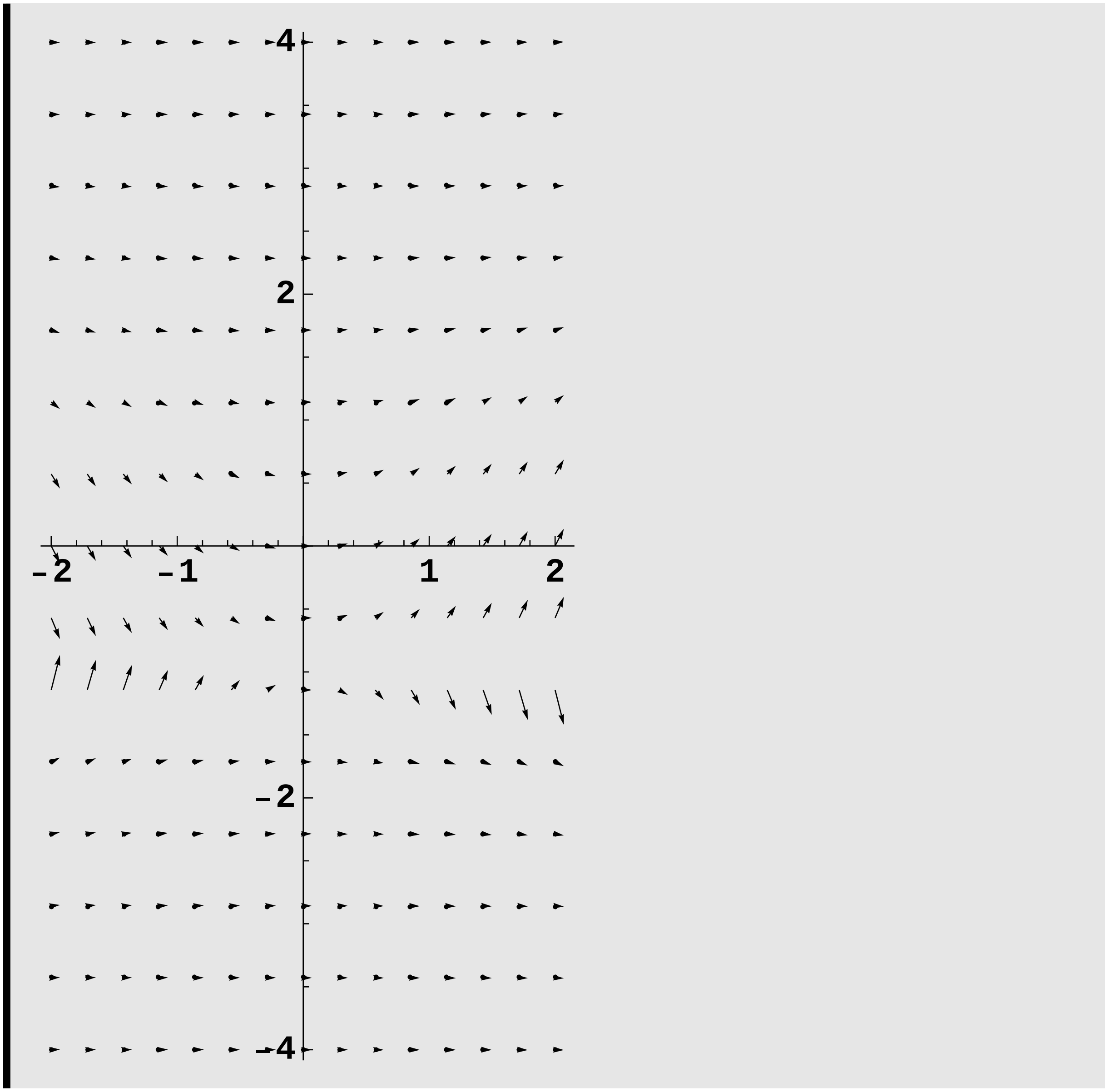


Note that we get a whole family of solution curves. Which

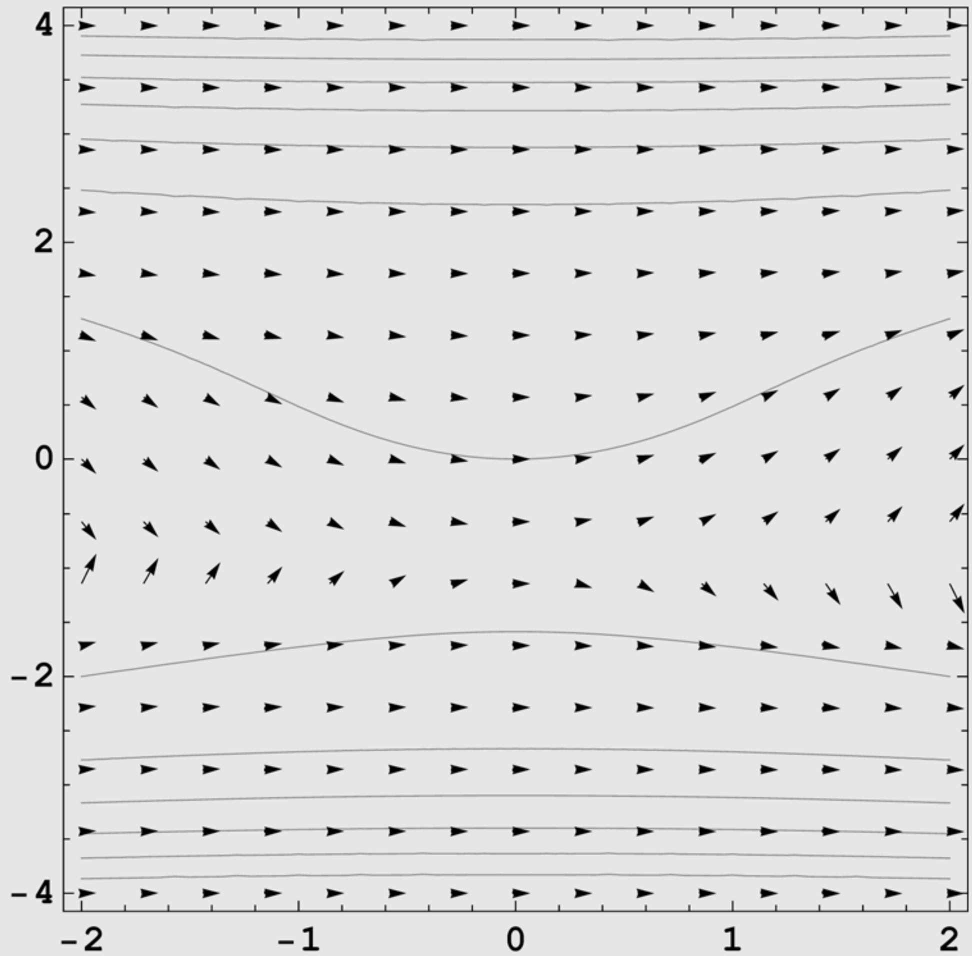
solution curve we want is defined by an initial condition.

```
vf = (Needs["VectorFieldPlots`"];
  VectorFieldPlots`VectorFieldPlot[{1, t/(1 + y^3)},
   {t, -2, 2}, {y, -4, 4}, Axes → True])
```

General::obspkg :
VectorFieldPlots` is now obsolete. The legacy version being loaded may conflict with current *Mathematica* functionality. See the Compatibility Guide for updating information. ≫


4
2
-2
-1
1
2
-2
-4

```
Show[cp, vf]
```



Example of Solution by Separation (2)

Solve the initial value problem

$$(1 + t)\ y'[t] = t\, y^2/(1 + t)^{1/2} \quad \text{with} \quad y(0) = 1.$$

By inspection, we can see that the differential equation is separable.

The idea is to force *Mathematica* to set up and solve the equation by mimicing what we do on paper. Set up the differential equation with a name:

```
Clear[ cp,de, y, t, implicitsoln]
```

```
de = Sqrt[1 + t] * D[y[t], t] == t*y[t]^2/Sqrt[1 + t]
```

$$\sqrt{1+t}\ y'[t] == \frac{t\, y[t]^2}{\sqrt{1+t}}$$

Divide through to put the equation in separated form :

Do this in your head or on scratch paper, there is no convenient way to force Mathemtica to do this operation!

```
(1/y[t]^2) D[y[t],t] == t/(1 + t)
```

$$\frac{y'[t]}{y[t]^2} == \frac{t}{1+t}$$

```
lhs = Integrate[ y'[t]/y[t]^2, t]
```

$$-\frac{1}{y[t]}$$

```
rhs = Integrate[t/(1 + t), t]
```

$$t - \text{Log}[1 + t]$$

```
implicitsoln = lhs == rhs + C
```

$$-\frac{1}{y[t]} == C + t - \text{Log}[1 + t]$$

```
Solve[ implicitsoln, y[t]]
```

$$\left\{\left\{y[t] \to \frac{1}{-C - t + \text{Log}[1 + t]}\right\}\right\}$$

```
implicitsoln /. { t -> 0, y[t] -> 1}
```

$$-1 == C$$

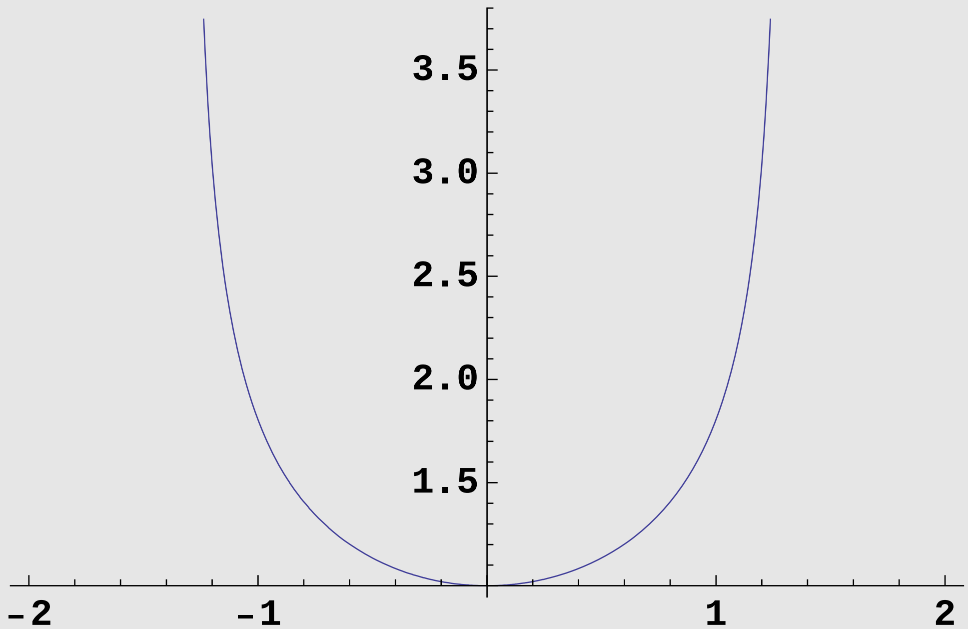
```
impsolnwithIC = implicitsoln /. { C -> -1/2}
```

$$-\frac{1}{y[t]} == -\frac{1}{2} + t - \text{Log}[1 + t]$$

```
Solve[ impsolnwithIC, y[t]]
```

$$\left\{\left\{y[t] \to -\frac{2}{-1 + 2\,t - 2\,\text{Log}[1 + t]}\right\}\right\}$$

```
Plot[ 1/(Sqrt[ 1- Log[1+t^2]]), {t, -2,2}]
```



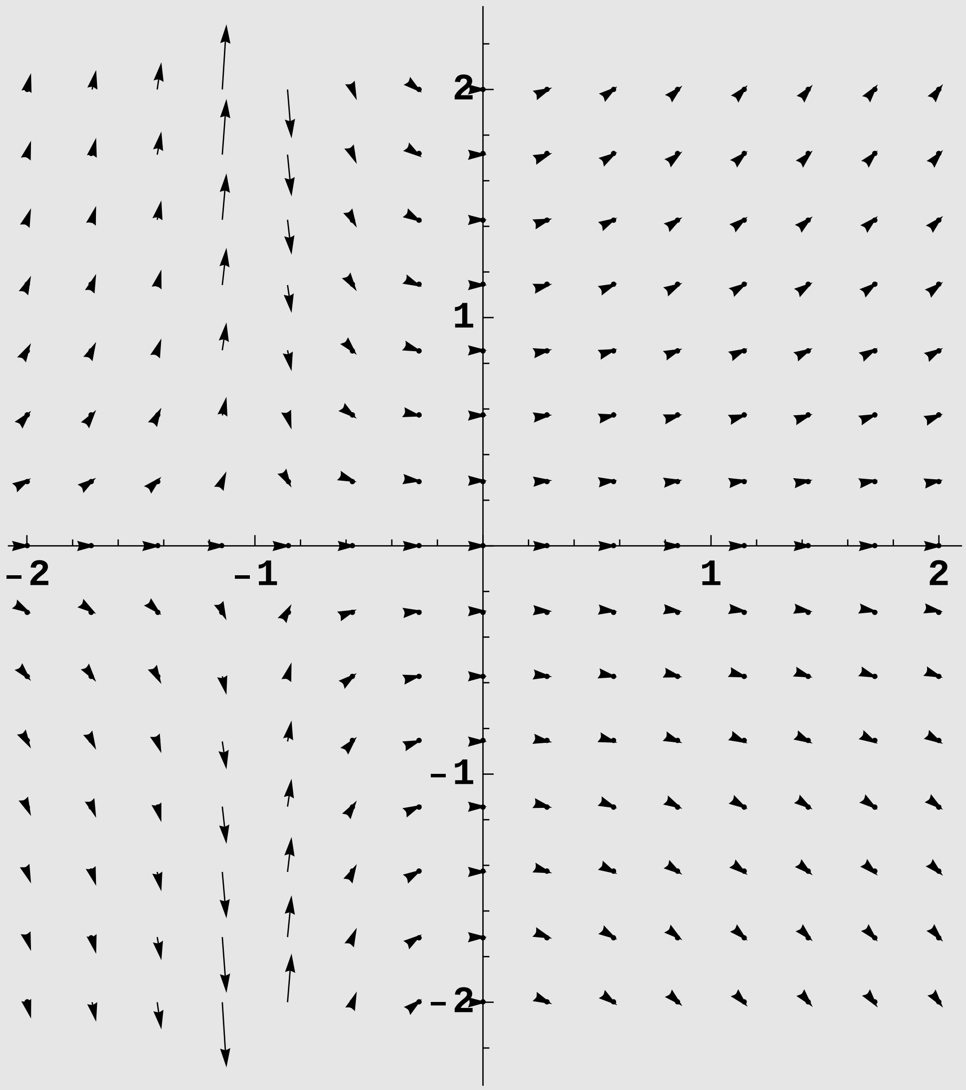

```
<< "VectorFieldPlots`"
```

General::obspkg :
VectorFieldPlots` is now obsolete. The legacy version being loaded may conflict with current *Mathematica* functionality. See the Compatibility Guide for updating information. ≫

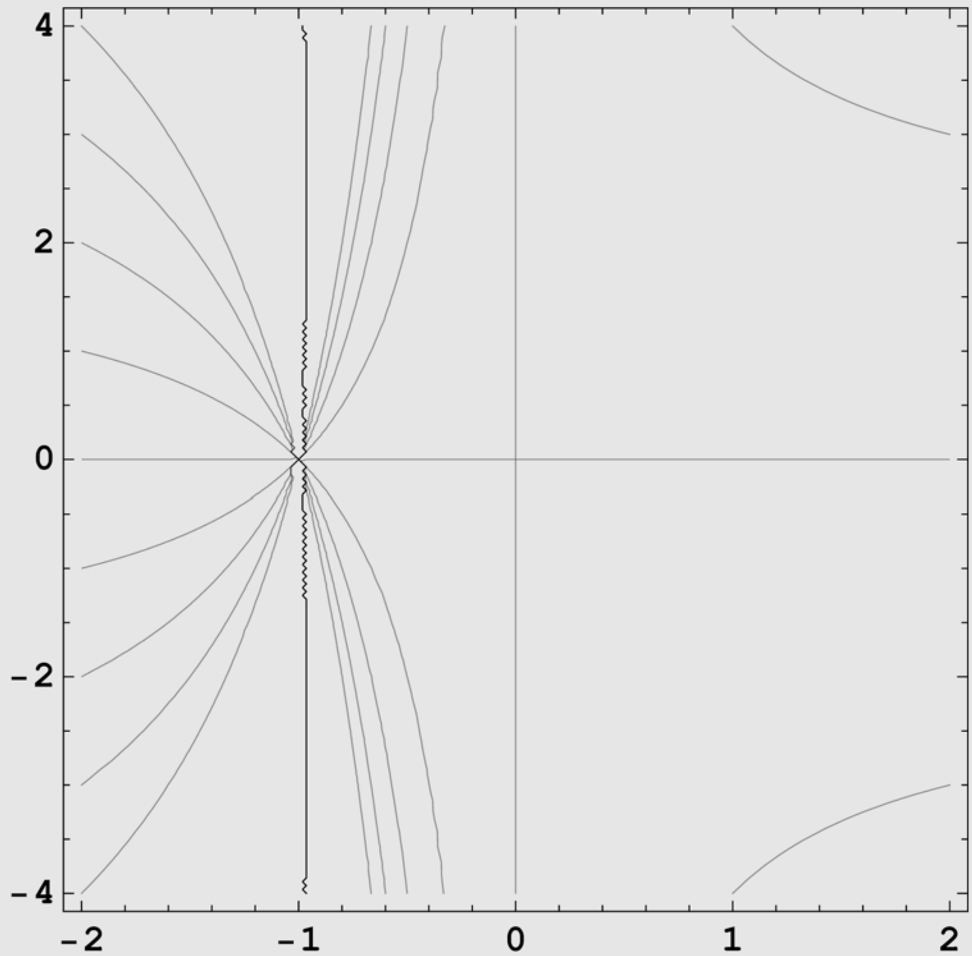
The normal form for the differential equation is

$$y'[t] = \frac{t\,y}{1 + t}$$

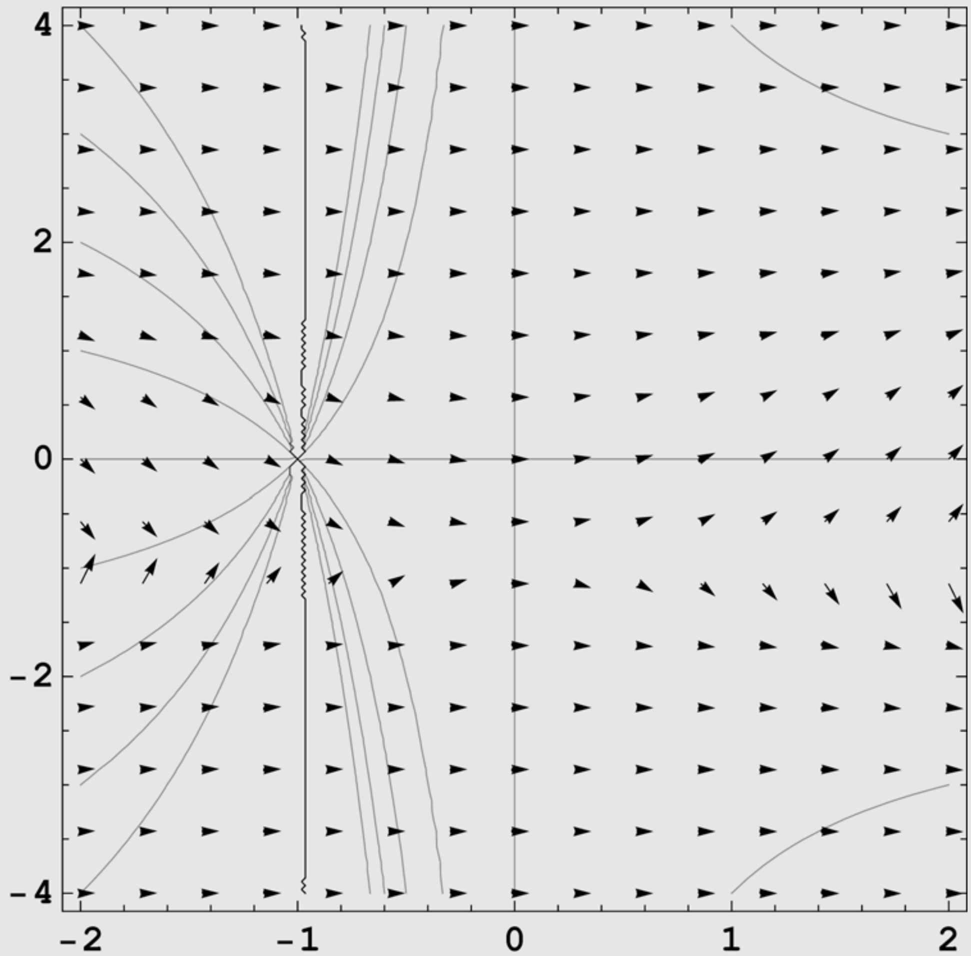
```
Needs["VectorFieldPlots`"];
VectorFieldPlots`VectorFieldPlot[{1, t y/(1 + t)},
  {t, -2, 2}, {y, -2, 2}, Axes → True]
```



```
cp = ContourPlot[t y/(1 + t), {t, -2, 2}, {y, -4, 4},
ContourShading -> False]
```



```
Show[cp, vf]
```



The normal form for the differential equation is

$$y'[t] = \frac{t\,y}{1+t}$$

Does the graph of the solution agree with what you guess from the direction fields?

```
?DSolve
```

DSolve[*eqn*, *y*, *x*] solves a differential equation for the function *y*, with independent variable *x*.
DSolve[{eqn_1, eqn_2, …}, {y_1, y_2, …}, *x*] solves a list of differential equations.
DSolve[*eqn*, *y*, {x_1, x_2, …}] solves a partial differential equation. ≫

```
DSolve[de, y[t],t]
```

$$\left\{\left\{y[t] \to \frac{1}{-t - C[1] + \text{Log}[1 + t]}\right\}\right\}$$

Differential Operators(2)

```
ClearAll["Global`*"];
Off[General::spell, General::spell1]
```

Define some Linear Constant Coefficient Differential Operators

$L_1[y] = y'' - y' + y$; $L_2[y] = y'' + 2\,y$; $L_3[y] = 2y'' + 5\,y' + 3\,y$

```
L1[f_] := D[f,{x,2}]-D[f,x] + f
```

```
L2[f_] := D[f, {x, 2}] + 2 f
```

```
L3[f_] := 2 D[f, {x, 2}] + 5 D[f, x] + 3 f
```

Some functions into these differential operators

```
L1[ x^2]
L2[x^2]
L3[x^2]
```

$2 - 2\,x + x^2$

$2 + 2\,x^2$

$4 + 10\,x + 3\,x^2$

```
L1[ Sin[c x]]
L2[ Sin[c x]]
L3[ Sin[c x]]
```

$-c\,\text{Cos}[c\,x] + \text{Sin}[c\,x] - c^2\,\text{Sin}[c\,x]$

$2\,\text{Sin}[c\,x] - c^2\,\text{Sin}[c\,x]$

$5\,c\,\text{Cos}[c\,x] + 3\,\text{Sin}[c\,x] - 2\,c^2\,\text{Sin}[c\,x]$

```
L1[ Exp[a x] Cos[b x] Sin[c x] ] // Simplify
L2[ Exp[a x] Cos[b x] Sin[c x] ] // Simplify
L3[ Exp[a x] Cos[b x] Sin[c x] ] // Simplify
```

$$-e^{a x}\left(b\,\mathrm{Sin}[b x]\,(2 c\,\mathrm{Cos}[c x]+(-1+2 a)\,\mathrm{Sin}[c x])+\mathrm{Cos}[b x]\left((c-2 a c)\,\mathrm{Cos}[c x]+\left(-1+a-a^2+b^2+c^2\right)\mathrm{Sin}[c x]\right)\right)$$

$$e^{a x}\left(-2 b\,\mathrm{Sin}[b x]\,(c\,\mathrm{Cos}[c x]+a\,\mathrm{Sin}[c x])+\mathrm{Cos}[b x]\left(2 a c\,\mathrm{Cos}[c x]+\left(2+a^2-b^2-c^2\right)\mathrm{Sin}[c x]\right)\right)$$

$$e^{a x}\left(-b\,\mathrm{Sin}[b x]\,(4 c\,\mathrm{Cos}[c x]+(5+4 a)\,\mathrm{Sin}[c x])+\mathrm{Cos}[b x]\left((5+4 a)\,c\,\mathrm{Cos}[c x]+\left(3+5 a+2 a^2-2 b^2-2 c^2\right)\mathrm{Sin}[c x]\right)\right)$$

Solutions of Second Order Linear Differential Equations

Example of solution and plotting

Use the Mathematica command NDSolve to numerically solve second order differnetial equations. NDSolve uses a very advanced and sophisticated method, similar in spirit to the improved Euler predictor-corrector method.

Our purpose is to solve :

3*y''[x] + 2*y'[x] + 7*y[x] == 0 ; y[0] == 1, y'[0] == -1

```
L[f_] := 3 * D[f, {x, 2}] + 2 * D[f, x] + 7 * f
```

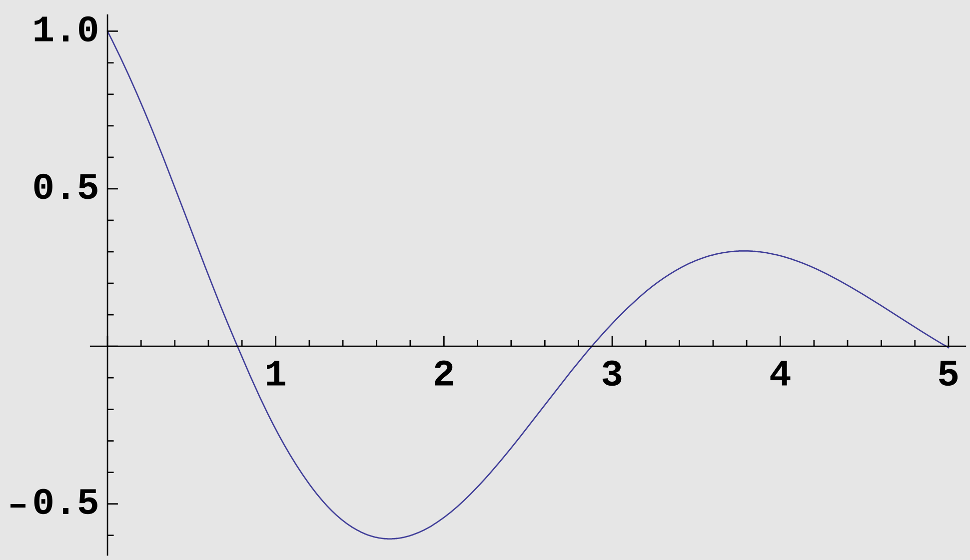
```
L[y[x]]
```

$7\,y[x] + 2\,y'[x] + 3\,y''[x]$

```
soln = NDSolve[{L[y[x]]== 0,
                   y[0] == 1, y'[0] == -1},
                   y[x], {x,0,10}]
```

```
{{y[x] → InterpolatingFunction[{{0., 10.}}, <>][x]}}
```

```
Plot[ y[x] /. soln, {x, 0, 5}, PlotRange -> All]
```



Experiments to Perform

(Borelli and Coleman, Experiment 4.2.3 , page 122) Consider the IVP

2*y'' - 5*y' + b*y == 0,   y(0) == 1, y'(0) == -1.

Graph y(x) for several values of b, (positive, negative, and zero). Are there any values of b for which the solution is oscillatory? What do you guess is the general form of the solutions?

```
ClearAll["Global`*"];
```

```
A[g_] := 2 * D[g, {x, 2}] - 5 * D[g, x] + b * g
```

```
A[y[x]]
```

```
b y[x] - 5 y′[x] + 2 y′′[x]
```

```
sol = DSolve[{A[y[x]] == 0, y[0] == 1, y '[0] == -1},
  y[x], x]
```

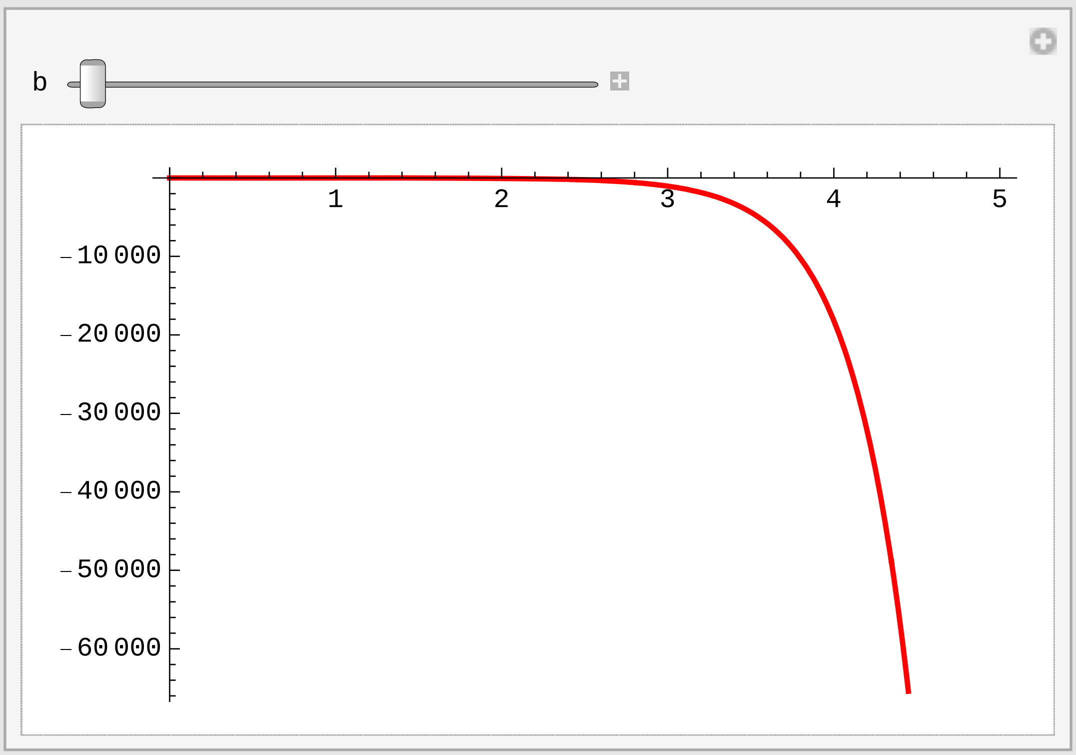
$$\left\{\left\{y[x] \to \frac{1}{2\sqrt{25-8b}}\left(9\, e^{\frac{1}{4}\left(5-\sqrt{25-8b}\right)x} + \sqrt{25-8b}\; e^{\frac{1}{4}\left(5-\sqrt{25-8b}\right)x} - 9\, e^{\frac{1}{4}\left(5+\sqrt{25-8b}\right)x} + \sqrt{25-8b}\; e^{\frac{1}{4}\left(5+\sqrt{25-8b}\right)x}\right)\right\}\right\}$$

```
y1[b_, x_] := y[x] /. Last[sol]
```

```
y1[b, x]
```

$$\frac{1}{2\sqrt{25-8b}}\left(9\,e^{\frac{1}{4}\left(5-\sqrt{25-8b}\right)x}+\sqrt{25-8b}\;e^{\frac{1}{4}\left(5-\sqrt{25-8b}\right)x}-9\,e^{\frac{1}{4}\left(5+\sqrt{25-8b}\right)x}+\sqrt{25-8b}\;e^{\frac{1}{4}\left(5+\sqrt{25-8b}\right)x}\right)$$

```
Manipulate[
 Plot[1/(2 Sqrt[25 - 8 b])
   (9 E^(1/4 (5 - Sqrt[25 - 8 b]) x) + Sqrt[25 - 8 b] E^(1/4 (5 - Sqrt[25 - 8 b]) x) - 9 E^(1/4 (5 + Sqrt[25 - 8 b]) x) +
     Sqrt[25 - 8 b] E^(1/4 (5 + Sqrt[25 - 8 b]) x)), {x, 0, 5}, PlotStyle → {Red, Thick}],
 {b, -2, 2}]
```

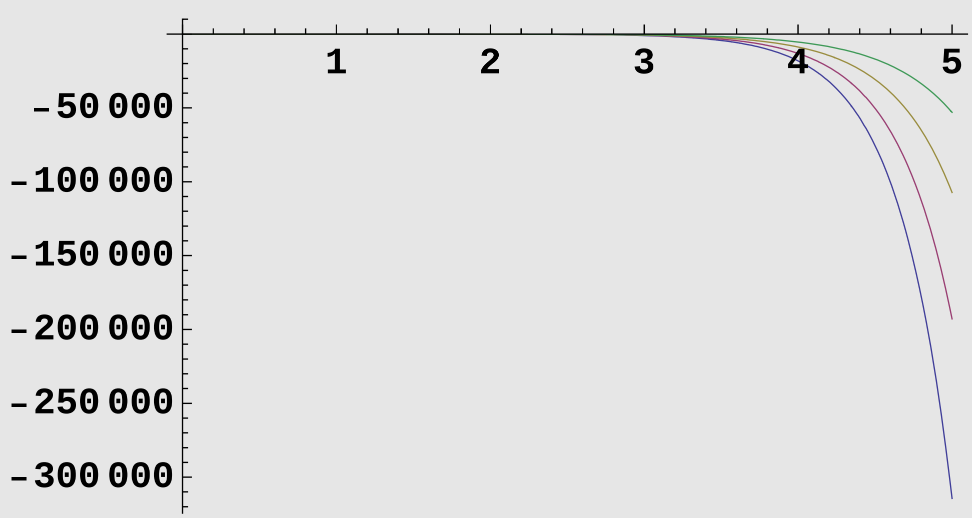


```
For[b = -2, b < 2, b++, Print[y[b, x]]]
```

```
y[-2, x]
y[-1, x]
y[0, x]
y[1, x]
```

p1 =

$$\text{Plot}\Big[\Big\{\frac{1}{2\sqrt{41}}\left(9\,e^{\frac{1}{4}\left(5-\sqrt{41}\right)x}+\sqrt{41}\,e^{\frac{1}{4}\left(5-\sqrt{41}\right)x}-9\,e^{\frac{1}{4}\left(5+\sqrt{41}\right)x}+\sqrt{41}\,e^{\frac{1}{4}\left(5+\sqrt{41}\right)x}\right),\ \frac{1}{2\sqrt{33}}\left(9\,e^{\frac{1}{4}\left(5-\sqrt{33}\right)x}+\sqrt{33}\,e^{\frac{1}{4}\left(5-\sqrt{33}\right)x}-9\,e^{\frac{1}{4}\left(5+\sqrt{33}\right)x}+\sqrt{33}\,e^{\frac{1}{4}\left(5+\sqrt{33}\right)x}\right),\ \frac{1}{10}\left(14-4\,e^{5x/2}\right),\ \frac{1}{2\sqrt{17}}\left(9\,e^{\frac{1}{4}\left(5-\sqrt{17}\right)x}+\sqrt{17}\,e^{\frac{1}{4}\left(5-\sqrt{17}\right)x}-9\,e^{\frac{1}{4}\left(5+\sqrt{17}\right)x}+\sqrt{17}\,e^{\frac{1}{4}\left(5+\sqrt{17}\right)x}\right)\Big\},\ \{x,0,5\},\ \text{PlotRange}\to\text{All}\Big]$$



Ex1: Try to obtain

y'' - a*y' - y == 0, y(0) == 1, y'(0) == -1.

Graph y(x) for several values of a, (positive, negative, and zero). Are there any values of a for which the solution is oscillatory? What do you guess is the general form of the solutions?

Applications-Mass-Spring Systems and Resonance

Comparing the effects of damping coefficients

An interesting problem is to compare the the effect of different values of the damping

coefficient c on the resulting motion of the mass on the spring. Consider the following problem: A 5 kg mass is attached to a spring with spring constant 8 newtons/meter. Determine the equation of motion which results if the initial displacement is y(0) = 6, the initial velocity is 0, there is no forcing, and the friction coefficient is

(a) ν = 0

(b) ν= 10

(c) ν = 15

In any case the differential equation is

$$5\, y''(t) + \nu y'(t) + 8\, y(t) = 0$$

$$y(0) = 6$$

$$y'(0) = 0$$

```
ClearAll["Global`*"];
Off[General::spell, General::spell1]
```

```
soleq1 = DSolve[ {5 y''[t] +0 y'[t] + 8 y[t] == 0,
                 y[0] == 6,
                 y'[0] == 0},
                 y[t],t]
```

$$\left\{\left\{y[t] \to 6 \cos\left[2 \sqrt{\frac{2}{5}}\, t\right]\right\}\right\}$$

```
y1 = soleq1[[1,1,2]]
```

$$6 \cos\left[2 \sqrt{\frac{2}{5}}\, t\right]$$

```
soleq2 = DSolve[ {5 y''[t] +10 y'[t] + 8 y[t] == 0,
                 y[0] == 6,
                 y'[0] == 0},
                 y[t],t]
```

$$\left\{\left\{y[t] \to 2\, e^{-t} \left(3 \cos\left[\sqrt{\frac{3}{5}}\, t\right] + \sqrt{15} \sin\left[\sqrt{\frac{3}{5}}\, t\right]\right)\right\}\right\}$$

```
y2 = soleq2[[1,1,2]]
```

$$2\, e^{-t} \left(3 \,\mathrm{Cos}\left[\sqrt{\frac{3}{5}}\; t\right] + \sqrt{15}\, \mathrm{Sin}\left[\sqrt{\frac{3}{5}}\; t\right]\right)$$

```
ComplexExpand[y2]
```

$$6\, e^{-t}\, \mathrm{Cos}\left[\sqrt{\frac{3}{5}}\; t\right] + 2\sqrt{15}\, e^{-t}\, \mathrm{Sin}\left[\sqrt{\frac{3}{5}}\; t\right]$$

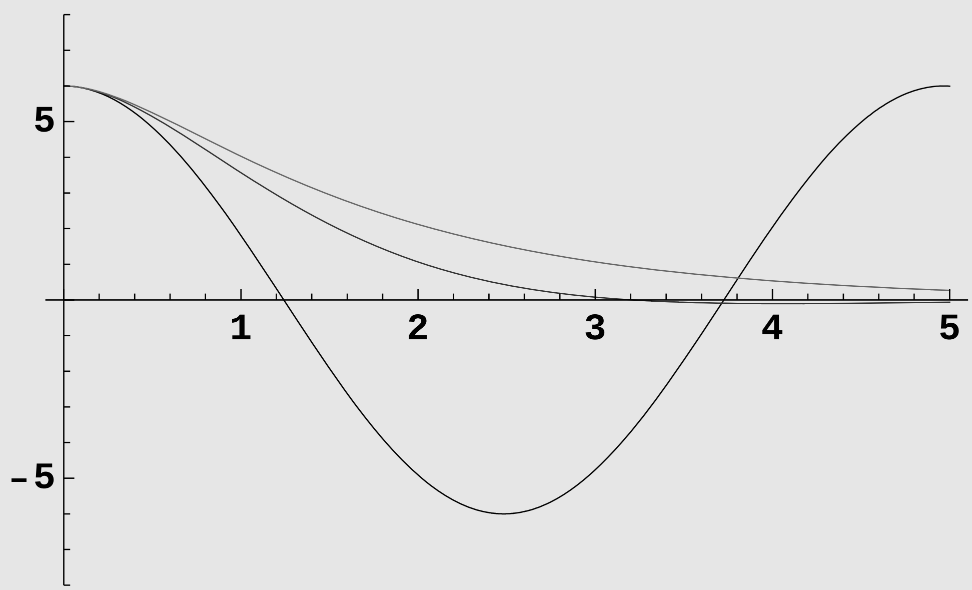
```
soleq3 = DSolve[ {5 y''[t] +15 y'[t] + 8 y[t] == 0,
                y[0] == 6,
                y'[0] == 0},
                y[t],t]
```

$$\left\{\left\{y[t] \to -\frac{3}{13}\left(-13\, e^{\left(-\frac{3}{2}-\frac{\sqrt{\frac{13}{5}}}{2}\right) t} + 3\sqrt{65}\, e^{\left(-\frac{3}{2}-\frac{\sqrt{\frac{13}{5}}}{2}\right) t} - 13\, e^{\left(-\frac{3}{2}+\frac{\sqrt{\frac{13}{5}}}{2}\right) t} - 3\sqrt{65}\, e^{\left(-\frac{3}{2}+\frac{\sqrt{\frac{13}{5}}}{2}\right) t}\right)\right\}\right\}$$

```
y3 = soleq3[[1,1,2]]
```

$$-\frac{3}{13}\left(-13\, e^{\left(-\frac{3}{2}-\frac{\sqrt{\frac{13}{5}}}{2}\right)t}+3\sqrt{65}\, e^{\left(-\frac{3}{2}-\frac{\sqrt{\frac{13}{5}}}{2}\right)t}-13\, e^{\left(-\frac{3}{2}+\frac{\sqrt{\frac{13}{5}}}{2}\right)t}-3\sqrt{65}\, e^{\left(-\frac{3}{2}+\frac{\sqrt{\frac{13}{5}}}{2}\right)t}\right)$$

```
Plot[ {y1,y2,y3}, {t,0,5},
    PlotRange -> { -8,8},
        PlotStyle -> {GrayLevel[0], GrayLevel[.2],
                GrayLevel[.4], GrayLevel[.6]}]
```



Resonance as a result of damping and forcing

We consider a forced mass-spring oscillator system with friction or damping. The mass is taken to be 2 kg. The spring constant is taken to be 6 newtons/meter. The damping can be varied. The frequency of the forcing term can also be varied. We want to look at the response of the system to the this forcing under a variety of damping conditions.

The equation is

y''[t] + ν y'[t] + 6 y[t] = 2 Sin[a t]

y[0] = 0

y'[0] = 0

The forcing frequency coefficient is a.

The damping coefficient is ν.

```
Clear[y,t]
```

```
homeq = 2*y''[t] + ν y'[t] + 6 y[t] == 0
```

$$6\, y[t] + \nu\, y'[t] + 2\, y''[t] == 0$$

```
DSolve[homeq, y[t], t]
```

$$\left\{\left\{y[t] \to e^{\frac{1}{4} t \left(-\nu-\sqrt{-48+\nu^2}\right)} C[1] + e^{\frac{1}{4} t \left(-\nu+\sqrt{-48+\nu^2}\right)} C[2]\right\}\right\}$$

```
y0 = y[t] /. %
```

$\left\{e^{\frac{1}{4} t \left(-\nu-\sqrt{-48+\nu^2}\right)} C[1] + e^{\frac{1}{4} t \left(-\nu+\sqrt{-48+\nu^2}\right)} C[2]\right\}$

```
y0 = First[%]
```

$e^{\frac{1}{4} t \left(-\nu-\sqrt{-48+\nu^2}\right)} C[1] + e^{\frac{1}{4} t \left(-\nu+\sqrt{-48+\nu^2}\right)} C[2]$

```
yp = A Cos[a t] + B Sin[ a t]
```

A Cos[a t] + B Sin[a t]

```
D[yp, {t,2}] + ν D[yp, t] + 6 yp
```

$-a^2 A \, Cos[a\, t] - a^2 B \, Sin[a\, t] + \nu (a B \, Cos[a\, t] - a A \, Sin[a\, t]) + 6 (A \, Cos[a\, t] + B \, Sin[a\, t])$

```
Collect[ %, {Cos[a t], Sin[a t]}]
```

$(6 A - a^2 A + a B \nu) \, Cos[a\, t] + (6 B - a^2 B - a A \nu) \, Sin[a\, t]$

```
uceqns = { 6 A - a^2 A + a B ν == 0, 6 B - a^2 B - a A
```

$\{6 A - a^2 A + a B \nu == 0, 6 B - a^2 B - a A \nu == 2\}$

```
coeffs = Solve[uceqns, {A, B}]
```

$$\left\{\left\{A \to -\frac{2\, a\, \nu}{36 - 12\, a^2 + a^4 + a^2\, \nu^2},\ B \to -\frac{2\left(-6 + a^2\right)}{36 - 12\, a^2 + a^4 + a^2\, \nu^2}\right\}\right\}$$

```
yp = First[ A Cos[ a t] + B Sin[ a t] /. coeffs]
```

$$-\frac{2\, a\, \nu\, \mathrm{Cos}[a\, t]}{36 - 12\, a^2 + a^4 + a^2\, \nu^2} - \frac{2\left(-6 + a^2\right) \mathrm{Sin}[a\, t]}{36 - 12\, a^2 + a^4 + a^2\, \nu^2}$$

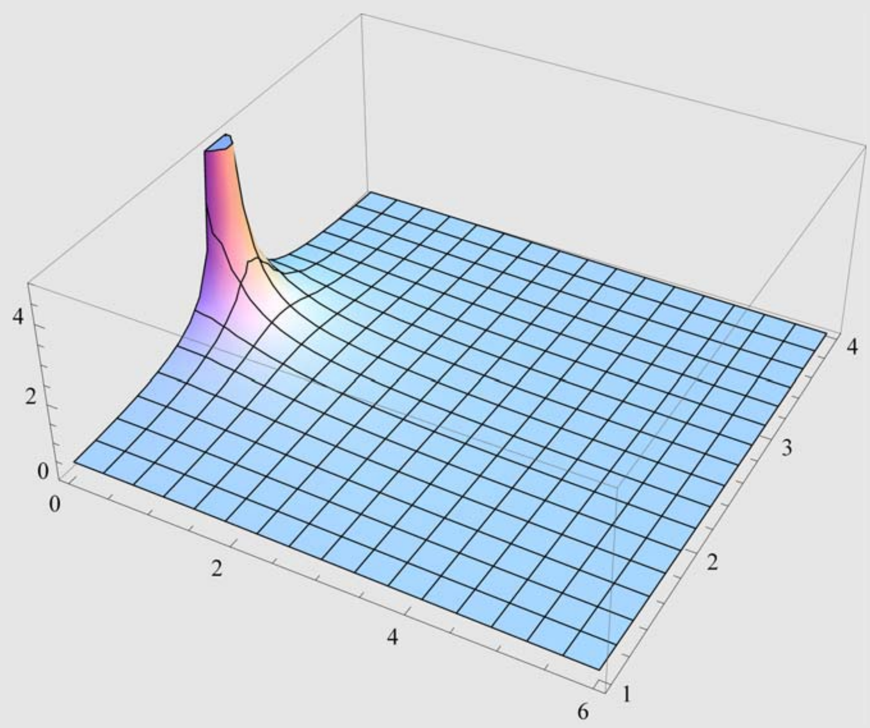
Having solved for the particular solution, I now put that particular solution in amplitude-phase form.

```
F0 = First[Sqrt[ A^2 + B^2] /. coeffs]//Simplify
```

$$2\sqrt{\frac{1}{36 + a^4 + a^2\left(-12 + \nu^2\right)}}$$

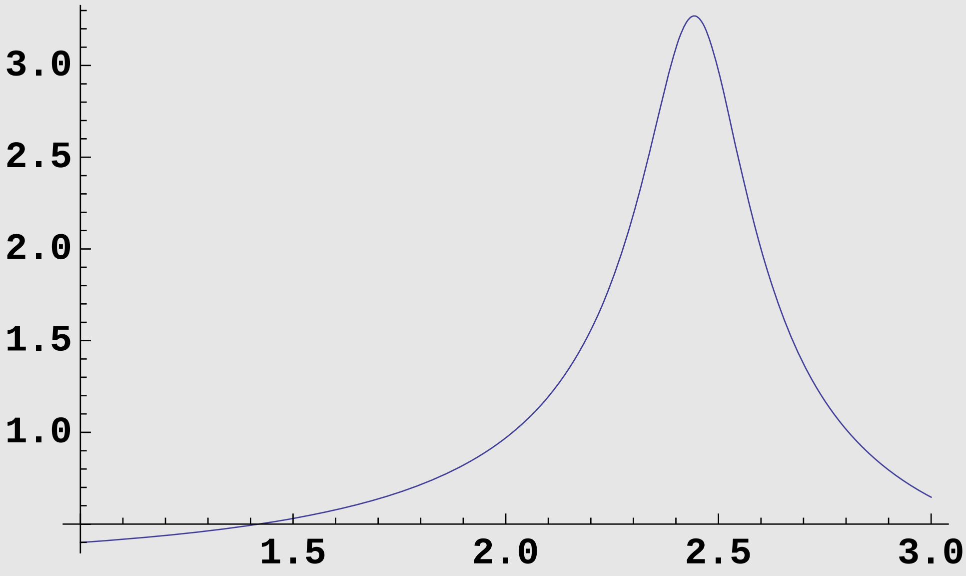
This then is the amplitude of the steady state response to the input forcing term. Now we wantt to plot the amplitude of the steady-state response as a function of the input frequency and the damping coefficient.

```
Plot3D[ F0, {ν, 0, 6}, {a, 1,4}, PlotRange -> {0, 5}]
```



Notice that something funny is happening near a = 2, and ν = 0. The error messages suggest that the graph "hits" inifinity near here. Notice too that the natural frequency of the undamped oscillator is 2! This sounds suspiciously like the dictionary definition given above! Lets plot the response as a function of forcing frequnecy for a damping coefficient near ν = 0.

```
Plot[F0 /. ν -> 0.25, {a, 1,3}]
```



The responce gets large near a = 2, and ν = 0.25. Let's actually find the full solution to the initial value problem.

```
y =(y0 + yp) /. {a -> 2, ν -> 0.25}
```

$$e^{(-0.0625-1.73092\, i)\, t}\, C[1] + e^{(-0.0625+1.73092\, i)\, t}\, C[2] - 0.235294\, Cos[2\, t] + 0.941176\, Sin[2\, t]$$

```
ComplexExpand[ y]
```

$$e^{-0.0625\, t}\, C[1]\, Cos[1.73092\, t] + e^{-0.0625\, t}\, C[2]\, Cos[1.73092\, t] - 0.235294\, Cos[2\, t] + i\left(-e^{-0.0625\, t}\, C[1]\, Sin[1.73092\, t] + e^{-0.0625\, t}\, C[2]\, Sin[1.73092\, t]\right) + 0.941176\, Sin[2\, t]$$

```
ic1 =( y /. t-> 0) == 0
```

$-0.235294 + C[1] + C[2] == 0$

```
ic2 = (D[y,t] /. t-> 0) == 0
```

$1.88235 - (0.0625 + 1.73092\, i)\, C[1] - (0.0625 - 1.73092\, i)\, C[2] == 0$

```
iceqns = { ic1, ic2}
```

$\{-0.235294 + C[1] + C[2] == 0,\ 1.88235 - (0.0625 + 1.73092\, i)\, C[1] - (0.0625 - 1.73092\, i)\, C[2] == 0\}$

```
Solve[ iceqns, {C[1], C[2]}]
```

$\{\{C[1] \to 0.117647 - 0.539495\, i,\ C[2] \to 0.117647 + 0.539495\, i\}\}$

```
y = y /. %
```

$\{(0.117647 - 0.539495\, i)\, e^{(-0.0625-1.73092\, i)\, t} + (0.117647 + 0.539495\, i)\, e^{(-0.0625+1.73092\, i)\, t} - 0.235294\, \text{Cos}[2\, t] + 0.941176\, \text{Sin}[2\, t]\}$

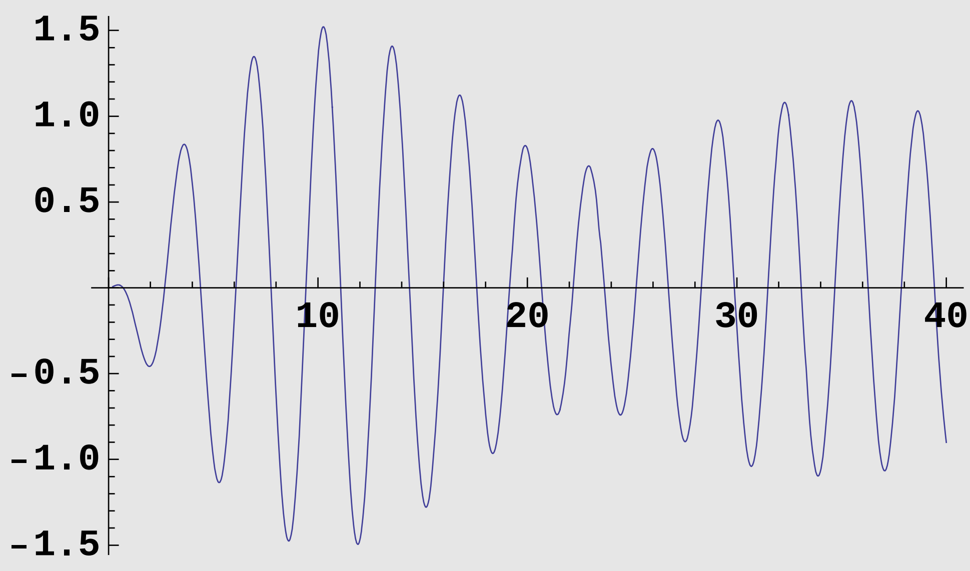
```
ComplexExpand[ First[%]]
```

$$0.235294\, e^{-0.0625\, t}\, \text{Cos}[1.73092\, t] - 0.235294\, \text{Cos}[2\, t] - 1.07899\, e^{-0.0625\, t}\, \text{Sin}[1.73092\, t] + i \left(-1.11022 \times 10^{-16}\, e^{-0.0625\, t}\, \text{Cos}[1.73092\, t] + 1.38778 \times 10^{-17}\, e^{-0.0625\, t}\, \text{Sin}[1.73092\, t]\right) + 0.941176\, \text{Sin}[2\, t]$$

```
trialsoln =Expand[%]
```

$$\left(0.235294 - 1.11022 \times 10^{-16}\, i\right) e^{-0.0625\, t}\, \text{Cos}[1.73092\, t] - 0.235294\, \text{Cos}[2\, t] - \left(1.07899 - 1.38778 \times 10^{-17}\, i\right) e^{-0.0625\, t}\, \text{Sin}[1.73092\, t] + 0.941176\, \text{Sin}[2\, t]$$

```
Plot[ trialsoln,{t, 0, 40},PlotPoints -> 50 ]
```



So the solution has been amplified ! The steady state amplitude is about double that of the input! Now let's see what happpens when there is no friction.

Forced, Undamped Motion

Let's see what happpens when there is no friction.

The equation has changed considerably, so we should properly re-solve from the beginning.

```
Clear[z,t]
```

```
hde = z''[t] + 6 z[t]  == 0
```

```
6 z[t] + z''[t] == 0
```

```
DSolve[hde, z[t], t]
```

$$\{\{z[t] \to C[1] \text{Cos}[\sqrt{6}\, t] + C[2] \text{Sin}[\sqrt{6}\, t]\}\}$$

```
z0 = %[[1,1,2]]
```

$$C[1] \text{Cos}[\sqrt{6}\, t] + C[2] \text{Sin}[\sqrt{6}\, t]$$

```
zp = C t Cos[2 t] + D t Sin[2 t]
```

```
C t Cos[2 t] + D t Sin[2 t]
```

```
D[ zp, {t,2} ] + 6 zp
```

```
4 D Cos[2 t] - 4 C t Cos[2 t] - 4 C Sin[2 t] -
  4 D t Sin[2 t] + 6 (C t Cos[2 t] + D t Sin[2 t])
```

```
Simplify[%]
```

$$2\,((2\,D + C\,t)\,\mathrm{Cos}[2\,t] + (-2\,C + D\,t)\,\mathrm{Sin}[2\,t])$$

```
zp = (-1/2) t Cos[ 2t]
```

$$-\frac{1}{2}\,t\,\mathrm{Cos}[2\,t]$$

```
D[ zp, {t,2}] + 6 zp
```

$$-t\,\mathrm{Cos}[2\,t] + 2\,\mathrm{Sin}[2\,t]$$

```
z = z0 + zp
```

$$-\frac{1}{2}\,t\,\mathrm{Cos}[2\,t] + C[1]\,\mathrm{Cos}\left[\sqrt{6}\,t\right] + C[2]\,\mathrm{Sin}\left[\sqrt{6}\,t\right]$$

```
ic1 = (z /. t -> 0) ==0
```

$$C[1] == 0$$

```
ic2 = (D[z,t] /. t -> 0) == 0
```

$$-\frac{1}{2} + \sqrt{6}\,C[2] == 0$$

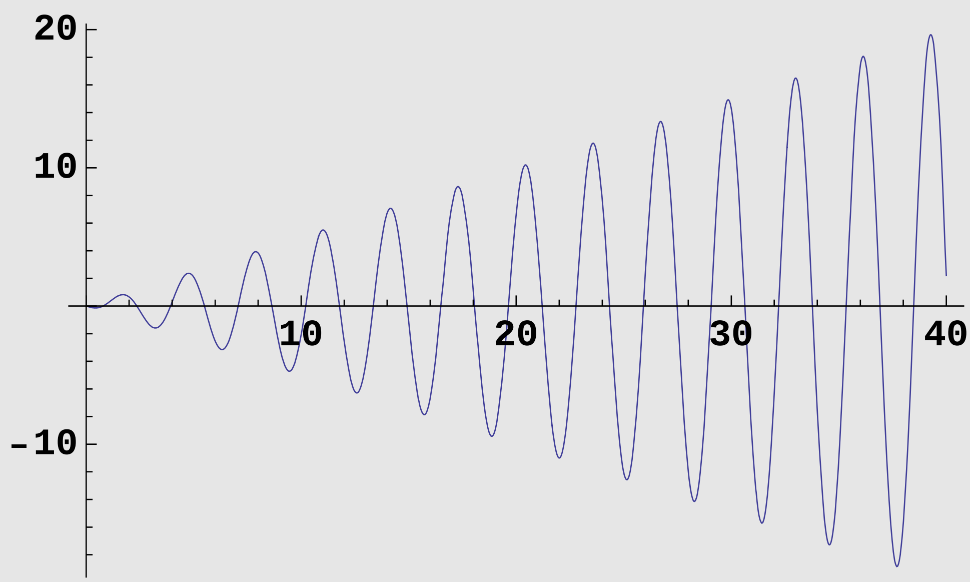
```
Solve[ {ic1, ic2}, {C[1], C[2]}]
```

$$\left\{\left\{C[1] \to 0,\ C[2] \to \frac{1}{2\,\sqrt{6}}\right\}\right\}$$

```
z = First[First[z /. %]]
```

$-\frac{1}{2} t \, Cos[2 t]$

```
Plot[ z, {t, 0, 40}, PlotPoints -> 50]
```



Notice that growing factor of t in the particualr solution. No wonder that the graphing complained about an infinite output response amplitude when the friction coefficient b = 0.! This is the case of pure resonance, the previous graph was the case of damped resonance.

Exercise

Given the initial value problem

$$x''(t) + x(t) = 2\cos\omega t + 3\text{sim}\,\omega t$$

$$x(0) = 0$$

$$x'(0) = 0$$

Find a value of ω such that the solution has beats, and graph the solution.

Find a value such that the solution is resonance, and graph the solution.

Find a value of ω such that the solution is the superposition of two periodic functions, the ratio of whose periods is irrational. Graph the solution. Is the solution periodic?